ON THE SINGULARITIES OF TORTUOUS CURVES

By Paul Saurel

The different singularities which a tortuous curve may present have been classified by von Staudt;* and Wiener,† by simple geometrical considerations, has determined the character of the projections of these singularities upon various planes. The analytical side of the question has also received attention and very complete discussions have been given by Fine,‡ by Staude,§ and by Meder.‖ But the results of these investigations have not yet been stated in the simplest terms. The following simple way of obtaining and of stating the analytical characteristics of the various types of singular points may accordingly be of interest.

1. We shall begin by considering a plane curve and we shall suppose that the coordinates x and y of a point on this curve are given by series of integral powers of a parameter t, convergent within a certain interval. We shall suppose, moreover, that to two different values of t within the interval there correspond two different points of the curve.

---

* *Geometrie der Lage*, p. 110; 1847.

† *Zeitschrift für Mathematik und Physik*, vol. 25, p. 95; 1880.

‡ *American Journal of Mathematics*, vol. 8, p. 156; 1886.

§ *American Journal of Mathematics*, vol. 17, p. 359; 1895.

‖ *Journal für die reine und angewandte Mathematik*, vol. 116, pp. 50, 247; 1896.

Let us take the point under consideration as the origin of coordinates and let us suppose that to this point corresponds the value zero of the parameter. The curve, in the neighborhood of the origin, is accordingly defined by the equations

$$x = at^p + \cdots, \qquad y = bt^q + \cdots, \tag{1}$$

in which the dots indicate higher powers of t, and a and b denote positive constants.

The straight line through the origin and the neighboring point t is given by the equation

$$\frac{x}{at^p + \cdots} = \frac{y}{bt^q + \cdots}. \tag{2}$$

As t approaches zero this line approaches as a limiting position the tangent to the curve at the origin. If we suppose the tangent at the origin to be the x axis, the above equation must approach the form $y = 0$. From this it follows at once that

$$p < q. \tag{3}$$

To classify the various kinds of singular points, von Staudt considers the motion of a point which describes the curve and the rotation of the tangent line which accompanies the tracing point. At any given point of the curve the tracing point may either continue its motion in the same direction or it may stop and begin to move in the opposite direction. The tangent also may either continue to rotate in the same direction about the tracing point or it may stop and begin to turn in the opposite direction. By combining each of the possible motions of the tracing point with each of the possible motions of the tangent line we obtain the following four types of points:

1. point continues, tangent continues;
2. point continues, tangent stops;
3. point stops, tangent continues;
4. point stops, tangent stops. (4)

If the tracing point continues its motion, x will change sign with t; p must therefore be odd. If, on the contrary, the tracing point stops, x will keep its sign while t changes from positive to negative; p must therefore be even.

If the tangent continues its rotation, the line joining the origin with a neighboring point on the curve will, for small positive values of t, pass through the first and third quadrants while, for small negative values of t, it will pass through the second and fourth quadrants. From equation 2 it follows that, in this case, $p + q$ will be odd. If, on the contrary, the tangent stops and begins to turn in the opposite direction, the line joining the origin with a neighboring point of the curve will pass through the first and third quadrants for small positive or negative values of t. From equation 2 it follows that, in this case, $p + q$ will be even.

We have accordingly the following analytical characteristics of the four types of points: *

$$p < q;$$

$$\begin{array}{lll} 1. & p \text{ odd}, & q \text{ even}; \\ 2. & p \text{ odd}, & q \text{ odd}; \\ 3. & p \text{ even}, & q \text{ odd}; \\ 4. & p \text{ even}, & q \text{ even}. \end{array} \tag{5}$$

The following equations are the simplest that can be written for each of the four classes of points. The accompanying sketches show the forms of the curves in the immediate neighborhood of the origin.

1. $x = t, \quad y = t^2;$ 

2. $x = t, \quad y = t^3;$  (6)

3. $x = t^2, \quad y = t^3;$ 

4. $x = t^2, \quad y = t^4 + t^5.$ 

2. We next consider a curve in space. If we take the point under consideration as the origin, and if we suppose that to this point corresponds the value zero of the parameter, we may write the equations of the curve in the form

$$x = at^p + \cdots, \quad y = bt^q + \cdots, \quad z = ct^r + \cdots, \tag{7}$$

in which the dots indicate higher powers of t, and a, b, c denote positive constants.

---

* *Cf.* Jordan, *Cours d'analyse*, vol. 1, p. 400; 1893.

The straight line through the origin and the neighboring point t is given by the equations

$$\frac{x}{at^p + \cdots} = \frac{y}{bt^q + \cdots} = \frac{z}{ct^r + \cdots}. \tag{8}$$

As t approaches zero, this line approaches as a limiting position the tangent to the curve at the origin. If we suppose the tangent at the origin to be the x axis, the above equations must reduce to $y = 0$, $z = 0$. From this it follows that

$$p < q, \qquad p < r. \tag{9}$$

The plane through the x axis and a point t is given by the equation

$$\frac{y}{bt^q + \cdots} = \frac{z}{ct^r + \cdots}. \tag{10}$$

As t approaches zero, this plane approaches as a limiting position the osculating plane at the origin. If we suppose the osculating plane at the origin to be the xy plane, the above equation must reduce to $z = 0$. From this it follows that

$$q < r. \tag{11}$$

To classify the various kinds of singular points, von Staudt considers the motion of a tracing point which describes the curve, the rotation in the osculating plane of the tangent which accompanies the point, and the rotation of the osculating plane about the tangent. At any given point, each of these elements, point, line, and plane, may either continue its motion in the same direction or may stop and begin to move in the opposite direction. There thus arise the following eight types of points:

1.	point continues,	tangent continues,	plane continues ;	
2.	point continues,	tangent continues,	plane stops ;	
3.	point continues,	tangent stops,	plane continues ;	
4.	point continues,	tangent stops,	plane stops ;	(12)
5.	point stops,	tangent continues,	plane continues ;	
6.	point stops,	tangent continues,	plane stops ;	
7.	point stops,	tangent stops,	plane continues ;	
8.	point stops,	tangent stops,	plane stops.	

If the tracing point continues its motion, x will change sign with t; p must therefore be odd. If, on the contrary, the tracing point stops, x will keep its sign while t changes from positive to negative; p must therefore be even.

If the tangent continues to rotate in the same direction, the line joining the origin with the point in the xy plane which is the projection of the point t of the curve will, for positive values of t, lie in the first and third quadrants of the xy plane, and for negative values of t in the second and fourth quadrants. From the first of equations 8 it follows that $p + q$ must be odd. If, on the contrary, the tangent stops and begins to turn in the opposite direction, the line in the xy plane will pass through the first and third quadrants for positive and negative values of t. Accordingly, in this case, $p + q$ must be even.

If the osculating plane continues to revolve in the same direction, so will the plane given by equation 10; so also will the intersection of this plane with the yz plane. From equation 10 it follows that $q + r$ must be odd. If, on the contrary, the osculating plane stops, equation 10 shows that $q + r$ must be even.

The characteristics of each of the eight types are accordingly given by the following table:

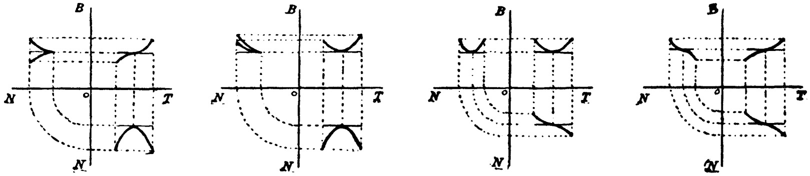
$$p < q < r;$$

1.	p odd,	q even,	r odd;	
2.	p odd,	q even,	r even;	
3.	p odd,	q odd,	r even;	
4.	p odd,	q odd,	r odd;	(13)
5.	p even,	q odd,	r even;	
6.	p even,	q odd,	r odd;	
7.	p even,	q even,	r odd;	
8.	p even,	q even,	r even.	

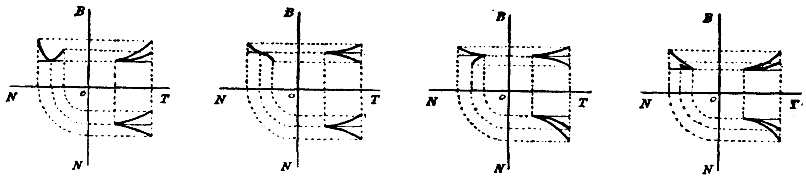
The equations 14, below, are the simplest that can be written for each of the eight kinds of points. By means of them the character of the projections of the singularities upon the coordinate planes can be determined. The accompanying sketches give the projections upon planes parallel respectively to the normal plane, the rectifying plane, and the osculating plane; the letters T, N, B indicate lines parallel respectively to the tangent, the principal

normal, and the binormal of the curve.

$$\begin{array}{llll} 1. & x = t, & y = t^2, & z = t^3; \\ 2. & x = t, & y = t^2, & z = t^4 + t^5; \\ 3. & x = t, & y = t^3, & z = t^4; \\ 4. & x = t, & y = t^3, & z = t^5; \end{array} \tag{14a}$$



$$\begin{array}{llll} 5. & x = t^2, & y = t^3, & z = t^4 + t^5; \\ 6. & x = t^2, & y = t^3, & z = t^5; \\ 7. & x = t^2, & y = t^4 + t^5, & z = t^5; \\ 8. & x = t^2, & y = t^4 + t^5, & z = t^6 + t^7. \end{array} \tag{14b}$$



Equations 14 also show that the eight types of singular points correspond to the eight possible ways in which a moving point, which has reached the origin from the first octant, may proceed; it may continue its path into any one of the eight octants.*

3. The preceding results can be extended to a space of any number of dimensions. If

$$x_i = a_i t^{p_i} + \cdots, \qquad i = 1, 2, \cdots, n, \tag{15}$$

be the equations of a curve in a space of n dimensions, and if $x_2 = x_3 = \cdots = x_n = 0$ be the equations of the tangent line, $x_3 = x_4 = \cdots = x_n = 0$ the

* Klein, *Anwendung der Differential- und Integralrechnung auf Geometrie*, p. 439; 1902.

equations of the osculating plane, $x_4 = x_5 = \cdots = x_n = 0$ the equations of the osculating space of three dimensions, and, in general, $x_{i+1} = x_{i+2} = \cdots = x_n = 0$ the equations of the osculating space of i dimensions, then $p_i < p_{i+1}$ and the motion of the osculating space of i dimensions persists or reverses according as $p_i + p_{i+1}$ is odd or even. Each type of singular points is characterized by the distribution of the odd and the even numbers in the series of p's, and the number of types is 2^n.

NEW YORK, MARCH, 1905.